Lecture 5:

Communications, Recurrence, Transience

Last Time

~~Stationary Distributions~~

$$\vec{\pi} = \vec{\pi} \cdot P \qquad \text{(Left eigenvector)}$$

Prob. vector Transition matrix

Key Questions:

Q1. When do stationary distributions exist?

Q2. Under Q1, when are they unique?

Q3. Suppose there exists a unique stationary distribution $\vec{\pi}$, given $X_0 \sim \vec{\mu}$, will it always true that

$$\vec{\mu}(n) = \vec{\mu} \cdot P^n \xrightarrow{n \to \infty} \vec{\pi} \ ?$$

1°. Negative Results.

Ex1. Let $\mathcal{X} = \{1, 2\}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{\pi} = (\frac{1}{2}, \frac{1}{2})$,

then $\vec{\pi} P = \vec{\pi}$. Let $\vec{\mu} = (1, 0)$, then

$$\vec{\mu}(n) = \vec{\mu} \cdot P^n = \begin{cases} (1, 0), & \text{if } n \text{ is even;} \\ (0, 1), & \text{if } n \text{ is odd.} \end{cases}$$

does not converges, which gives a negative result to Q3.

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Ex2. Let $\mathcal{X} = \{1, 2\}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$S_{stat} := \{\vec{\pi} = \vec{\pi} P\}$$

$$= \{\text{all probability vectors}\}$$

$$= \{\pi = a(1,0) + b(0,1) \mid a+b=1, 0 \le a \le 1\}.$$

This example provides a negative answer to Q2.

2°. Def. A state x communicates with a state y if there exists $n \geq 1$, such that $[P^n]_{xy} > 0$.

Notation: $x \longrightarrow y$. (Note: Resnick does it differently.)

Def. First return/hitting time to x is defined as

$$\tau_x = \min\{n \geq 1 \mid X_n = x\}.$$

Let $\rho_{xy} = \mathbb{P}(\tau_y < \infty \mid X_0 = x) =: \mathbb{P}_x(\tau_y < \infty)$

starting at x.

be the probability of hitting y in finite steps, provided the initial state is x.

Lemma 1. $x \longrightarrow y$ if and only if $\rho_{xy} > 0$.

Pf. "$\Rightarrow$" Suppose $x \longrightarrow y$, then there exists $n \geq 1$, such that $[P^n]_{xy} > 0$.

Since $\{X_n = y\} \subseteq \{\tau_y < \infty\}$, we have

$$\mathbb{P}(X_n = y \mid X_0 = x) \leq \mathbb{P}(\tau_y < \infty \mid X_0 = x).$$

That is, $[P^n]_{xy} \leq \rho_{xy}$.

Since $[P^n]_{xy} > 0$, we have $\rho_{xy} > 0$.

"$\Leftarrow$". Suppose $\rho_{xy} > 0$.

Notice that $\{\tau_y < \infty\} = \bigcup_{k=1}^{\infty} \{X_k = y\}$.

Thus,
$$\begin{aligned}\mathbb{P}(\tau_y < \infty \mid X_0 = x) &= \mathbb{P}\Big(\bigcup_{k=1}^{\infty} \{X_k = y\} \mid X_0 = x\Big) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P}(\{X_k = y\} \mid X_0 = x) \\ &= \sum_{k=1}^{\infty} [P^k]_{xy}.\end{aligned}$$

From our assumption, $\mathbb{P}(\tau_y < \infty \mid X_0 = x) = \rho_{xy} > 0$, thus there exists $n \geq 1$, such that $[P^n]_{xy} > 0$.

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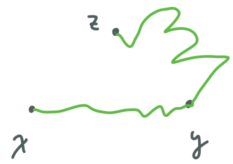
Lemma 2. (Transitivity) $x \longrightarrow y,\ y \longrightarrow z \Rightarrow x \longrightarrow z.$



Pf. Since $x \longrightarrow y$, $\exists\, m \geq 1$, s.t. $[P^m]_{xy} > 0$;

$y \longrightarrow z$, $\exists\, n \geq 1$, s.t. $[P^n]_{yz} > 0$.

Thus,
$$
\begin{aligned}
[P^{m+n}]_{xz} &= [P^m \cdot P^n]_{xz} \\
&= \sum_{w \in \mathcal{X}} [P^m]_{xw} [P^n]_{wz} \\
&\geq [P^m]_{xy} [P^n]_{yz} \\
&> 0.
\end{aligned}
$$

Therefore, $x \longrightarrow z$. $\blacksquare$

3°. Def. A state x is *transient* if $\rho_{xx} < 1$.

A state x is *recurrent* if $\rho_{xx} = 1$.

Ex3. (Gambler's Ruin) The transition matrix

$$
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

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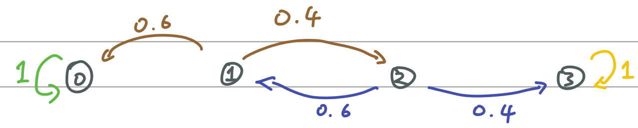

Q: Determine which states are transient/recurrent?

A: $\rho_{00} = \mathbb{P}_0(\tau_0 < \infty) = 1$.

$\rho_{33} = \mathbb{P}_3(\tau_3 < \infty) = 1$.

Therefore, the states 0 and 3 are recurrent.

Notice that $\{\tau_1 = \infty\} \supseteq \{X_1 = 0\}$.

$$1 - \rho_{11} = \mathbb{P}_1(\tau_1 = \infty) \geq \mathbb{P}(X_1 = 0 \mid X_0 = 1) = 0.6 > 0.$$

Thus, $\rho_{11} < 1$ and the state 1 is transient.

Similarly, the state 2 is also transient.

Remark 1. Notice that $[P^2]_{11} > 0$. Thus $1 \longrightarrow 1$ and $\rho_{11} > 0$. In this example, $0 < \rho_{11} < 1$.

Remark 2. In general, $x \longrightarrow y$ does not imply $y \longrightarrow x$.

e.g. In Ex 3, $\rho_{10} > 0$ but $\rho_{01} = 0$.

why? why?

Ex4. (3 coffee shops). Suppose the transition matrix

$$P = \begin{array}{c} \\ T \\ M \\ S \end{array} \begin{array}{c} \begin{array}{ccc} T & M & S \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

Q: Determine which states are transient/recurrent?

A: For any states x @ time n,

$$\mathbb{P}(X_{n+1} = S \mid X_n = x) \geq 0.1.$$

Thus, $\mathbb{P}(X_{n+1} \neq S \mid X_n = x) \leq 0.9$. This implies

$$\mathbb{P}(\tau_S > n \mid X_0 = S) \leq (0.9)^n. \qquad (*)$$

Notice that $\{\tau_S > k\} \supseteq \{\tau_S > k+1\}$, $\forall k \in \mathbb{N}$, and

$\{\tau_S = \infty\} = \{\tau_S > k \text{ for all } k \in \mathbb{N}\} = \bigcap_{k \geq 1} \{\tau_S > k\}$.

Note: Suppose $A := \bigcap_{n=1}^{\infty} A_n$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, then

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Taking limits at both sides of $(*)$ yields

$$\mathbb{P}_S(\tau_S = \infty) = \lim_{n\to\infty} \mathbb{P}_S(\tau_S > n) \leq \lim_{n\to\infty} (0.9)^n = 0.$$

Since $\mathbb{P}_S(\tau_S = \infty) \geq 0$, we have $\mathbb{P}_S(\tau_S = \infty) = 0$

and $\rho_{SS} = \mathbb{P}_S(\tau_S < \infty) = 1 - \mathbb{P}_S(\tau_S = \infty) = 1$.

Similarly, $\rho_{TT} = 1$ and $\rho_{MM} = 1$.

4°. **Theorem 1.** If $x \to y$ and $\rho_{yx} < 1$, then x is transient.

Pf. ①. Case I: If $y = x$, then $\rho_{xx} = \rho_{yx} < 1$.

By definition, x is transient.

②. Case II: If $y \neq x$, let

$$K := \min\{k \geq 1 \mid [P^k]_{xy} > 0\}.$$

Because $x \longrightarrow y$, we have $[P^K]_{xy} > 0$.

Notice that $[P^K]_{xy} = [\underbrace{P \cdot P \cdot \cdots \cdot P}_{K \text{ occurrence of "}P\text{"}}]_{xy}$

$$= \sum_{z_1, \cdots, z_{K-1} \in \mathcal{X}} P_{xz_1} \cdot P_{z_1 z_2} \cdot \cdots \cdot P_{z_{K-1} y}.$$

Therefore, there exists $x_1, x_2, \cdots, x_{K-1} \in \mathcal{X}$, such that

$$P_{xx_1} P_{x_1 x_2} \cdot \cdots \cdot P_{x_{K-1} y} > 0. \qquad (**)$$

Besides, $x_i \neq x$, $\forall x \in [K-1]$. (Otherwise, K is not the smallest integer k to make $[P^k]_{xy} > 0$, why?)

Notice that $\{T_x = \infty\} = \{X_n \neq x, \forall n \in \mathbb{N}\}$

$$\supseteq \{X_1 = x_1, X_2 = x_2, \cdots, X_{K-1} = x_{K-1}, X_K = y, X_n \neq x, \forall n > K\}$$

Thus, $\mathbb{P}_x(\tau_x = \infty)$

$$\geq \mathbb{P}_x(X_1 = x_1, X_2 = x_2, \cdots, X_k = y,\ X_n \neq x,\ \forall n > k)$$

$$= \mathbb{P}(X_n \neq x,\ \forall n > k \mid X_k = y, \cdots, X_0 = x) \cdot$$

$$\cdot\, \mathbb{P}(X_k = y,\ X_{k-1} = x_{k-1}, \cdots, X_1 = x_1 \mid X_0 = x_0)$$

$$= \mathbb{P}(X_n \neq x,\ \forall n > k \mid X_k = y) \cdot$$

$$\cdot\, \mathbb{P}(X_k = y \mid X_{k-1} = x_{k-1}) \cdot \mathbb{P}(X_{k-1} = x_{k-1} \mid X_{k-2} = x_{k-2}) \cdots$$

$$\cdot\, \mathbb{P}(X_1 = x_1 \mid X_0 = x)$$

$$= \mathbb{P}_y(\tau_x = \infty) \cdot P_{x_{k-1}y} \cdot P_{x_{k-2}x_{k-1}} \cdot \cdots \cdot P_{x_1x_2} P_{xx_1}$$

$$= (1 - \rho_{yx}) \cdot P_{xx_1} \cdots P_{x_{k-1}y}$$

$$> 0.$$

$\mathbb{P}(A, B \mid C) = \mathbb{P}(A \mid B, C) \cdot \mathbb{P}(B \mid C)$

Markov property

(**) $\rho_{yx} < 1$

Therefore, $\rho_{xx} = \mathbb{P}_x(\tau_x < \infty) = 1 - \mathbb{P}_x(\tau_x = \infty) < 1$.

By definition, x is transient. ▨

Corollary. If $x \longrightarrow y$ and x is recurrent, then $\rho_{yx} = 1$.

This is the end of this lecture!

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